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CNRS
CREA–Ecole Polytechnique
Paris, France

Pierre Kerszberg
Université de Toulouse-Le Mirail
Toulouse, France

Jean Petitot
Ecole des Hautes Etudes en Sciences
Sociales and
CREA–Ecole Polytechnique
Paris, France

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Can the Principle of Least Action Be Considered a Relativized A Priori?

Michael Stöltzner

Abstract Hardly another principle of classical physics has to a larger extent nourished hopes into a universal theory and has simultaneously been plagued by mathematical counterexamples than the Principle of Least Action (PLA). I investigate whether the PLA can be interpreted as a historically relativized constitutive a priori principle of mathematical physics along the lines Michael Friedman has drawn in *Dynamics of Reason*, using the example of relativity theory. Such an interpretation suggests itself, historically, because two main advocates of the PLA, Max Planck and David Hilbert, considered relativity theory as a case in point for the PLA. But they were also aware of the mathematical pitfalls and that without physical specification the PLA only represented an empty form. I argue that the different levels required for a consistent application of the PLA in mathematical physics induce a stratification that bears close parallels to the one by which Friedman intends to overcome the joint challenges of epistemological holism and a relativist reading of Kuhnian incommensurability. Yet, two differences remain. First, the mathematical and physical levels of the PLA are more intertwined than in Friedman's case. Second, although the PLA has survived quite a few scientific revolutions, so has the formulation of physical theories in terms of differential equations.

If one wants to embark on a Kantian analysis of the Principle of Least Action (henceforth PLA), there are basically two routes. First, one may focus on the formal teleology the PLA has often been associated with and interpret the PLA in the perspective of the *Critique of Judgment* as a regulative principle of reflective judgment. Along this line, as I have argued elsewhere (Stöltzner, 2000, 2005), the distinctive features of the PLA are: (i) its globality as compared to the local differential equations; (ii) the systemic modal structure of the actual dynamics and the possible dynamics that are set up in order to obtain a mathematically well-defined extremal principle. The strength of a formal-teleological surplus of the PLA over the standard

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M. Stöltzner
Department of Philosophy, University of South Carolina, Columbia, SC 29208, USA
e-mail: stoeltzn@sc.edu

formulation in terms of differential equations crucially hinges on mathematical subtleties and the concept of causal explanation assumed, rather than express a metaphysical harmony of the world.

Second, one may focus on the PLA's role as a mathematical principle that, after appropriate specification, permits one to succinctly formulate physical theories as different as classical mechanics, electrodynamics, relativity theory, and quantum physics. Taken in its abstract generality, the PLA has survived many vicissitudes of the scientific world-view and the minor or major revolutions in physical theory. This suggests viewing the PLA as a historically relativized constitutive a priori principle in the sense recently advocated by Michael Friedman (2001). Such is the line taken by the present paper. Friedman's conception of the dynamics of reason reaches back to the interpretation of the Kantian categories of space and time outlined in Hans Reichenbach's (1920) early analysis of the theory of relativity and, thus, to neo-Kantian debates about a modernization of the *Critique of Pure Reason* in the face of twentieth century physics.

Both assessments of the PLA do not necessarily contradict one another. On the one hand, the Marburg neo-Kantians, most prominently Ernst Cassirer, considered the Kantian categories not as constitutive but as regulative principles precisely to allow for a historical evolution of the basic principles of science in which the a priori appeared as an absolute – not historically relativized – invariant attained only in the ideal limit of the scientific enterprise as a whole. On the other hand, in the case of a mathematical principle applied to physics, the regulative principle of formal teleology represents a norm imposed on the mathematical architecture of a physical theory. If the PLA incorporating this formal teleology acts as the core mathematical axiom of a physical theory so conceived, it may thus be considered as constitutive for the particular natural laws – the mathematical models – derived from this axiom system. In a historical perspective, one might perhaps say that the first approach departs from Kant's critical analysis of the debates on physical teleology and natural theology that surrounded the birth of the PLA in the eighteenth century, while the second is associated with two important debates in the second half of the nineteenth century. Hermann von Helmholtz not only successfully applied the PLA beyond the narrower context of mechanics, thus bringing about its renaissance among his fellow physicists, but he also gave the 'return to Kant' prevailing within the German philosophical community an influential scientific twist.

Helmholtz's scientific achievements prompted the question whether the PLA represented "a valuable heuristic principle and leitmotif in striving for a formulation of the laws of new classes of phenomena" (1886, p. 210) or whether it was it just – as Ernst Mach held – a more economical reformulation of the same facts and thus "new only in *form* and not in *matter*" (1989, p. 452) Through the mathematical investigations of Karl Weierstraß and David Hilbert, it became clear that any surplus of the PLA over the differential equations required mathematical precision rather than strong metaphysical commitments. Nonetheless, Logical Empiricists largely sided with Mach, while Max Planck and Hilbert firmly believed in the significance of the PLA for a unified conception of physics. Even more so, Planck and Hilbert's emphatic pronouncement played an important role in making the PLA a shibboleth for Logical Empiricists (Stöltzner, 2003). This basic disagreement makes the PLA an interesting test case as to whether the Logical Empiricist conception of constitutive principles in natural science, which basically takes them as part of pragmatics, needs to be revised in the direction outlined by Friedman, in particular if one wants to understand the progress of modern mathematical physics.

After a brief characterization of the PLA and a rehearsal of Friedman's approach, I study Planck and Hilbert's interpretations of the PLA. Although both are in many respects water on Friedman's mills – and stand in historical vicinity to his primary example general relativity –, two major problems remain. First, the mathematical and physical levels of the PLA are more intertwined than Friedman assumes, and what counts as 'natural' or 'deep' according to the respective standards does not always coincide. Second, although the PLA has survived quite a few scientific revolutions, so has the formulation of physical theories in terms of differential equations. Hence, there have always been two different lines of constitutive principles that show little sign of convergence despite the fact that in many cases both formulations yield physically equivalent results.

1 The PLA in a Nutshell

For the purpose of the present paper, I understand the PLA as an umbrella term for all integral variational principles in theoretical physics, among them Hamilton's and Maupertuis' principles. Mathematicians treat all those principles within the discipline of variational calculus and speak of a variational problem rather than the PLA. In an abstract sense the PLA states that the actual dynamics u yields an extremal value of the action functional $W[u]$ in comparison to all possible dynamics $(u + \delta u) \in M_u$, where δ denotes the variation of a quantity, $u + \delta u$ the varied dynamics, and M_u a function space that includes both the actual and all possible dynamics. $W[u]$ is the integral of the Lagrangian L that incorporates the physics contained in the PLA. Since the PLA is an integral principle, the boundary conditions must be specified in order to arrive at a mathematically well-defined variational problem.

In classical mechanics, L is the difference of kinetic and potential energy, the PLA reads $W = \int_a^b L(t,u(t),\dot{u}(t))dt = \text{Extr.!}$, with δu, $\delta\dot{u} = 0$ at the boundary, and u belongs to a class of admitted solutions M ($=C^2, PC^1, \ldots$) of the variational problem. Variation leads to the Euler-Lagrange equations,

$$\frac{d}{dt}\frac{\partial}{\partial\dot{u}}L(t,u(t),\dot{u}(t)) - \frac{\partial}{\partial u}L(t,u(t),\dot{u}(t)) = 0$$

which typically – but not always – correspond to the standard equations of motion. But they are only a necessary condition for the variational problem. Other necessary conditions include the continuity properties of u, the form of the constraints imposed on the system, and that there are no (conjugate) points between a and b through which all $(u + \delta u) \in M_u$ have to pass. To find sufficient conditions has been a most difficult task. The first one was obtained only by Weierstraß, and Hilbert's main achievements in variational calculus lay precisely in this domain. Roughly speaking, sufficient conditions correspond to embedding the extremals into a suitable field of extremals (Mayer fields). This embedding expresses the above-mentioned global features of the PLA; and if a sufficient condition is fulfilled, the PLA represents a stronger claim than the corresponding differential equations.

2 The Dynamics of Reason

Friedman's overall intention is to develop a modified Kantian epistemology that successfully answers the combined challenges of W.V. Quine's epistemological holism and the prevailing relativist readings of Thomas S. Kuhn's theory of scientific revolutions. Friedman's analysis departs from diagnosing a far-reaching resemblance between Rudolf Carnap's (1950) linguistic frameworks and Kuhn's (1962) scientific paradigms. For, both frameworks and paradigms provide the basic concepts and rules under which science is 'normally' performed and define a standard of scientific rationality. The transitions between different frameworks or paradigms that characterize scientific revolutions cannot be assessed in this way, even though the later framework typically permits one to reconstruct the pre-revolutionary paradigm as a limit case and justify the revolution as rational. But this retrospective reconstruction fails to reflect the historical situation before the revolution, when the new concepts to come were still unavailable to science. Instrumentalistically-minded practitioners may well use bits and pieces of a new paradigm as a black-box device for making predictions, however without ascribing to it explanatory value. Predictive success alone, however, cannot motivate shifting badly understood or even ill-defined concepts into the core of a new paradigm or positing them as the axioms of a new linguistic framework, let alone overcome the problem of Kuhnian incommensurability.

The only remedy, Friedman argues, is to understand the scientific enterprise not as a succession of radically distinct speech communities but as "a common tradition of cultural change" that contains "different evolutionary stages of a single language" (2001, p. 60). This single language does not correspond to a single all-encompassing Carnapian framework, but contains stratifications that allow for communication across and transition between different frameworks or paradigms and render the new paradigm a real possibility already before the revolution. This "is already more than half the battle" (ibid. p. 103).

Friedman's dynamics of reason involves three different strata: (i) empirical laws properly so-called; (ii) a set of constitutive a priori principles that (a) render these laws meaningful as mathematical entities and (b) relate these mathematical entities to possible empirical circumstances by coordinative definitions; (iii) philosophical meta-principles or meta-paradigms that serve as a guidance "in motivating and sustaining the transition from one paradigm or conceptual framework to another" (ibid., p. 46). How the first two strata relate in detail can best be seen at the examples of Newtonian mechanics, relativistic electrodynamics, and general relativity, while

other disciplines are still lacking the degree of mathematization presupposed in Friedman's conception.

> [A]dvanced theories in mathematical physics … should be viewed as consisting of two asymmetrically functioning parts: a properly empirical part containing laws such as universal gravitation, Maxwell's equations of electromagnetism, or Einstein's equations for the gravitational field; and a constitutively a priori part containing both the relevant mathematical principles used in formulating the theory (Euclidean geometry, the geometry of Minkowski space—time, the Riemannian theory of manifolds) and certain particularly fundamental physical principles (the Newtonian laws of motion, the light principle, the equivalence principle) (ibid., p. 71).

Mathematical concepts are, for one, a condition of the possibility of physical theories, e.g., by allowing one to represent space-time in terms of Riemannian manifolds. But additionally pure mathematics has the remarkable property that across revolutions it usually tends to preserve the earlier concepts as a special case, such that in retrospect the new concepts appear as extensions or generalizations of the earlier ones. Euclidean geometry is, for instance, simply a Riemannian geometry with zero curvature. "Revolutionary transitions within pure mathematics, then, have the striking property of continuously (and, as it were, monotonically) preserving … *retrospective* communicative rationality" (ibid., p. 96). But mathematics alone cannot mediate across revolutions in physical science because the very same mathematical structures may be used to conceptualize incommensurable theories. Riemannian manifolds, for instance, are the mathematical basis of general relativity, but supplemented with a different set of coordinating principles they allow a mathematically elegant, albeit non-standard, reformulation of Newtonian mechanics.

It is the coordinative principles through which the formulas of mathematical physics acquire an empirical meaning. These principles cannot be tested in the usual sense because they are constitutive for those specific empirical laws that face the tribunal of experience. But they do have empirical content and are revisable in the course of history, although a scientific revolution is needed to unseat a well-established coordinative principle because of the failure of the particular laws constituted by it. The examples of the Michelson-Morley and the Eötvös experiment show that coordinative principles, such as the light principle and the principle of equivalence, often emerge out of well-corroborated empirical facts. But in elevating them to a constitutive a priori principle, "an essentially non-empirical element of 'convention' or 'decision' must necessarily intervene" (ibid., p. 91). Once the new paradigm is established and the old laws are reformulated in terms of it, the decision between the old and the new may appear as a plainly empirical fact.

The purely mathematical and the coordinative constitutive a priori principles, accordingly, not only define a space of logical and empirical possibility for physical science, but in virtue of their internal dynamics and mutual relationship they also suggest what counts as reason or justification for any such possibility. But even if a transformation can be justified retrospectively, the question remains how the new paradigm can at all develop from within the pre-revolutionary framework. Here the philosophical principles play a decisive role because they smoothen out the revolutionary transitions prospectively. To allow for a rational transition despite a Kuhnian incommensurability at the manifest level of the language spoken by the practitioners (cf. 101), Friedman requires

> first, that the new conceptual framework or paradigm should contain the previous constitutive framework as an approximate limiting case ...; second that the new constitutive principles should also evolve continuously out of the old constitutive principles, by a series of natural transformations; and third, that this process of continuous conceptual transformation should be motivated and sustained by an appropriate new philosophical framework, which, in particular, interacts productively with both older philosophical meta-frameworks and new developments taking place in the sciences themselves. This new philosophical meta-framework thereby helps to define what we mean, at this point, by a natural, reasonable, or responsible conceptual transformation (ibid., p. 66).

Moreover, philosophy provides "a new source of ideas, alternative programs, and expanded possibilities.(ibid., p. 25)". Philosophy can fulfill both roles only if it does not limit itself to a mere logic of science that is always bound to a rigid conceptual framework. But then one may wonder how "a subject inevitably and permanently fraught with unresolved intellectual disagreements [can] possibly help us to achieve a new rational consensus." But Friedman only requires: first, "that the new constitutive framework become a reasonable and responsible live option"; second, that there exists "a relatively stable consensus on what are the important contributions to the debate"; third, that "characteristically philosophical reflection interacts with properly scientific reflection in such a way that controversial and conceptually problematic philosophical themes become productively intertwined with relatively uncontroversial and unproblematic scientific accomplishments" (All ibid., p. 107).

In the case of special relativity theory, for instance, there was common agreement that Mach's criticism of the Newtonian concept of absolute motion was pivotal and combined with other investigations about relative and inertial motions. Einstein succeeded in putting these insights together with "recently established empirical facts concerning the velocity of light in a striking and hitherto unexpected manner" (ibid., p. 108) because he had been familiar with late nineteenth century debates on the foundations of geometry and Poincaré's conventionalist resolution of the problem how to determine the proper physical geometry. The same philosophical meta-principles also played an important, though somewhat different role, in general relativity that showed that even philosophical principles as deeply entrenched as space and time can undergo transitions.

Friedman's dynamics of reason also contains a global perspective. Putting together that, in virtue of the mathematical constitutive principles, transitions between different paradigms correspond to well-defined conceptual extensions and that the constitutive philosophical meta-principles provide the inter-paradigm transitions with a measure of naturalness, he argues "that we can thus view the evolution of succeeding paradigms or frameworks as a convergent series, as it were, in which we successively refine our constitutive principles in the direction of ever greater generality and adequacy" (ibid., p. 63). But "this is explicitly not convergence to an entirely independent 'reality' (however conceived) but rather convergence *within* the evolving sequence of constitutive frameworks itself" (ibid, p. 118), that is, 'internal' or Kant's 'empirical' realism. For, any strong version of scientific realism presupposes a unique sequence of successor theories, which contradicts the obvious plurality of historical pathways.

3 Max Planck on Principles and Constants

In 1915 Max Planck wrote an encyclopedia entry on the PLA. It opened emphatically.

> As long as there exists physical science, its highest desirable goal had been the solution of the problem to integrate all natural phenomena observed and still to be observed into a single simple principle .... It is natural that this goal has not been reached to date, nor ever will it be reached entirely. [However,] ... the history of theoretical physics demonstrates that ... this ideal problem is not merely utopical, but eminently fertile ... Among the more or less general laws which manifest the achievements of physical science in the course of the last centuries, the Principle of Least Action is probably the one which, as regards form and content, may claim to come nearest to that final ideal goal of theoretical research (1944, p. 68).

Planck was well aware of the mathematical pitfalls. Only after a precise mathematical specification of the Lagrangian and of the conditions for the virtual displacements the PLA ceased to be "an empty form" (ibid., p. 70). Moreover, when emphasizing that the PLA did not reintroduce any material teleology into physics but was consistent with a causal explanation of all natural phenomena, Planck took a surprisingly instrumentalist tack and compared the reference to events at a later time in the PLA to calculations in which one keeps redundant variables in order to maintain the symmetry of the equations. In both cases, "[t]he question of their legitimacy has nothing to do with teleology, but it is merely a practical one" (ibid., p. 72). And Planck even provided examples how the PLA led science astray if interpreted as instance of a universal teleology.

"The fundamental importance of the Principle of Least Action became generally recognized only when it proved its applicability to such systems whose mechanism is either completely unknown or too complex to think of a reduction to ordinary coordinates" (ibid., p. 76). For, the PLA as an integral principle was independent of any choice of coordinates. Around 1910, Planck became increasingly convinced that his law of black-body radiation required a fundamental break with classical electrodynamics because the latter unavoidably yielded Jeans's law, in blatant contradiction to everyday experience.

> [O]ne will not for this purpose have to give up the Principle of Least Action, which has so strongly attested its universal significance, but the universal validity of the Hamiltonian differential equations; for those are derived from the Principle of Least Action under the assumption that all physical processes can be reduced to changes occurring continuously in time. Once radiation processes do no longer obey the Hamiltonian differential equations, the ground is cut from Jeans's theory (1910, p. 239).

The PLA was not simply applicable to discontinuous functions as well, such functions had even been an important source of mathematical progress in variational calculus. The PLA was thus perfectly consistent with a different coordinative principle according to which atomic processes were quantized.

That the PLA was deeper than a merely heuristic principle can also be seen at the fact that it matched what Planck considered as the most basic distinction in the physical world. There were, on the one hand, reversible processes governed by strictly causal dynamical laws. "All of them can be subsumed without difficulty under a single dynamical law, the Principle of Least Action" (ibid., p. 59). "In the realm of irreversible processes, however, the Principle of Least Action is no longer sufficient because the principle of entropy increase introduces an entirely novel element into the physical world view that is in itself extraneous to the action principle" (ibid., p. 11).

Its pivotal status in the architecture of the physical world-view and the emphasis that the PLA represented a form to be completed by physical specifications suggest investigating whether Planck in effect treated the PLA as a relativized a priori in Friedman's sense. Since he did not attribute much importance to the mathematical architecture involved, we have to look whether setting up a suitable PLA represented a coordinative a priori principle on a par with Newton's laws or the principle of equivalence, while the respective Lagrangian corresponded to the empirical laws specific for each domain. Although the PLA had not developed out of well-known empirical facts, Planck had cited quantum discontinuity in its favor.

The main test for relativized constitutive principles is how they behave 'normally' and in times of turmoil. Here Planck was pretty explicit. "[I]n all recent conflicts [between facts and theories] the great general physical principles held the field, namely, the principle of conservation of energy, the principle of conservation of momentum, the Principle of Least Action, the main laws of thermodynamics" (ibid., p. 44), while well-accustomed intuitive foundations had to give way, among them the immutability of chemical atoms, the independence of space and time, and the continuity of all dynamical effects in nature. This insight was part of a general tendency of simultaneous de-anthropomorphization and unification that Planck diagnosed within modern physics. Moreover, our present picture already contained certain traits that most likely would remain constant forever. "This constancy ... is that which we now call the real [*das Reale*]"(ibid., p. 22). In 1925, Planck even conveived relativity theory within an overall convergence to absolute reality. "Yet when space and time have been denied the character of being absolute, the absolute has not been blotted out, it has just been moved more backward, to wit, into the metric of the four-dimensional manifold" (ibid., p. 154). Outdated absolute concepts are relativized just in order to find deeper absolute concepts.

Planck's convergent realism seems to openly contradict any relativized a priori. But Planck himself intended to remain consistent with Kant's critical philosophy. Since there was no way to distinguish between 'world view' and 'world', he argued, we could interpret 'world' itself as the ideal aim of all scientific research. Moreover, the constant elements in our world-view were abstract principles, such as the PLA, that remained empty without physical specification while entities such as the indivisible chemical atoms were superseded by new ones.

And there was also another side to Planck's realism. His own quantum of action and Boltzmann's constant characterizing thermal radiation, and the gravitational constant provide a universal system of units that does not depend on a metric convention. While present-day physicists aspire at reducing the fundamental constants of nature to laws, Planck considered them as "the invariable building blocks from which the edifice of the physical world is composed" (ibid., p. 39). Each major step towards the ideal aim of absolute knowledge uncovered a hitherto unknown constant of nature. First, "[t]he modification brought into mechanics by the principle of relativity contains as its essential part the introduction of a new universal constant alien to classical mechanics, the velocity of light in vacuum" (ibid., p. 82). Second, "the laws of thermal radiation, specific heat, electron emission, radioactivity unanimously indicate that not only matter itself but also the effects originating from matter ... possess discontinuous properties, which once again is characterized by a new constant of nature: the elementary quantum of action (ibid., p. 83f.).

Since the abstract principles held the field in each scientific revolution, it is the universal constants that represent a Kuhnian incommensurability, e.g., between the paradigms of classical mechanics and relativity theory or between Jeans's and Planck's laws of radiation. For, the fundamental constant of the latter theory cannot be expressed in terms of the former. If one formulates the latter theory by way of a PLA, the new constant, of course, has to enter. Thus the full-fledged action principle contains both constitutive and empirical elements.

To conclude, Planck treated the PLA in its abstract form as constitutive for the domain of reversible physics, while the coordination was established through a suitable Lagrangian. No doubt, Planck was well-informed about and derived major motivations from the philosophical debates surrounding the PLA, giving them a rather Kantian twist both as regards causality and formal teleology. Needless to say, Planck's talk about the ideal aim of theoretical physics did not exclude that the PLA one day could be integrated into a more comprehensive formal principle.

4 David Hilbert and the PLA as Core of the Axiomatic Method

Again in 1915, David Hilbert gave an independent derivation of the field equations of general relativity by means of a single action principle. The paper that was rushed out in two parts (1916, 1917) bore the ambitious title "The Foundations of Physics." In it, he combined what he had learned about the physical characteristics of Einstein's theory in the making with his top expertise in variational calculus. The correct form of Einstein's equations was entered only into the galley proofs, such that Hilbert cannot claim full priority. But they were not his primary objective. What is more, both "Hilbert and Einstein saw their achievements of November 1915 as the culmination of year-long efforts of scientific research along their respective research programs" (Sauer, 1999, p. 566). These were by no means identical, followed different heuristics, and attributed different weight to the PLA (Rowe, 2001).

In the "Foundations", Hilbert poses four axioms and two physicality conditions. (I) Mie's axiom of the world function H demands that the variation of $\int H\sqrt{g}\,d\omega$ vanishes for each gravitational potential $g_{\mu\nu}$ and each electromagnetic potential q_s, where g is the determinant of $g_{\mu\nu}$ and $d\omega = d\omega_1 d\omega_2 d\omega_3 d\omega_4$ is the differential of the world parameters ω_k uniquely fixing the world points. H contains gravitational arguments, the $g_{\mu\nu}$ and their first an second partial derivatives with respect to the ω_k, and electromagnetic arguments, the q_s and their first partial derivatives with respect to the ω_k. Axiom (II) states that H be invariant with respect to an arbitrary transformation of the world parameters ω_k. Hilbert considers this axiom as "the simplest mathematical expression of the requirement that the coordinates in themselves do not possess any physical significance" (1924, p. 50). And in a footnote, he connects it to Einstein's idea of general invariance (today: 'covariance'). Hilbert next formulates a theorem that he calls the "leitmotif for the construction of his theory" (1916, p. 396), but does not provide a proof. Its main objective had been to show that "*the electrodynamic phenomena are an effect of gravitation*" (1916, p. 397). In the 1924 reprint, this claim was tacitly dropped and a weakened version of Noether's second theorem was proven as theorem II.

Although sufficient for a derivation of geometrical properties, such as Noether's second theorem and the Bianchi identities, axioms (I) and (II) do not fix H uniquely, such that Hilbert introduced two further axioms of a more physical kind. Axiom (III) demands the additivity of pure gravity and electromagnetism $H = R + L$, with R being the Riemann scalar curvature and L not containing second derivatives of the $g_{\mu\nu}$. This guarantees that no higher than second order derivatives of the $g_{\mu\nu}$ appear in the field equations, such that one obtains a reasonable dynamics. Axiom (IV) specifies the signature of the metric in order to obtain the required 3 + 1 pseudo-geometry for space-time. In addition, there are two supplementary conditions requiring that the physical solutions respect causal order and are free of singularities. Gödel's universe, in which one can return into one's own past, and the boom of research into singularities since the 1960s have shown that both conditions of Hilbert's were too restrictive.

Hilbert's axiomatization of general relativity exhibits a three-layered structure that not only goes naturally with the different steps in specifying the PLA, but that is also typical for his axiomatic method as a whole. Hilbert did not treat an axiom system as a homogeneous conceptual framework in which only logical deductions operate. This can already be seen in the Sixth Problem of 1900, where he gave a programmatic outline of the axiomatization of physics.

> [W]e shall try first by a small number of axioms to include as large a class as possible of physical phenomena, and then by adjoining new axioms to arrive gradually at the more special theories .... The mathematician will have also to take account not only of those theories coming near to reality, but also, as in geometry, of all logically possible theories.
>
> Further, the mathematician has the duty to test exactly in each instance whether the new axioms are compatible with the previous ones. The physicist, *as his theories develop*, often finds himself forced by the results of his experiments to make new hypotheses, while he depends, with respect to the compatibility of the new hypotheses with the old axioms, solely upon these experiments or upon a certain physical intuition (1900, p. 272f./454f.).

The task of the mathematician begins with establishing the completeness of the axioms, i.e., that they permit one to derive all laws of the respective domain. Next is the internal consistency of the axioms. For instance, in the theory based on Fourier's heat equation "it is necessary to prove that the familiar boundary-value problem of potential theory is always solvable; for only the solution of this boundary-value problem shows that a temperature distribution satisfying the equation of heat conduction is at all possible" (1918, p. 410/1111). Cast in Friedman's terms, internal consistency of a suitable axiom system is an a priori condition that a certain physical phenomenon is logically possible.

By interpreting an axiom system in terms of appropriate number fields, Hilbert played internal consistency back to the consistency of arithmetic, which was to be proven by the finite means of meta-mathematics. But in 1930, Gödel's incompleteness theorem demonstrated that it was impossible to reach the desired absolute foundation of mathematics along these lines. Interestingly, in the same year, Hilbert viewed precisely this attempt as the legitimate heir of the Kantian a priori that "still contains anthropological dross from which it must be liberated [such as the preference of Euclidean geometry]; afterwards only the *a priori* attitude is left over which also underlies pure mathematical knowledge: essentially it is the finite attitude which I have characterized in several works" (1930, p. 962/1163). Hilbert's third requirement was external consistency. Kinetic theory is consistent with thermodynamics, and Einsteinian gravity possesses a well-defined Newtonian limit, while quantum theory contradicts Maxwell's equations, such that a new foundation of electrodynamics is called for. Recall that Friedman saw the main role of mathematics in establishing such a precise relationship between conceptual frameworks.

The fourth requirement, which Hilbert called 'deepening the foundations', started from the analysis of the independence of the axioms and was an heir of the failed attempts to prove the fundamental presuppositions of science themselves. But these reductions "only make it possible to trace things back to certain deeper propositions, which in turn now to be regarded as new axioms .... The actual *axioms* of geometry, arithmetic, statics, mechanics, radiation theory, or thermodynamics arose in this way" (1918, p. 407/1109). There was no unique way of 'deepening the foundations' of a given theory. Hilbert lauded both Boltzmann and Hertz for having deepened the foundations of Lagrange's mechanics containing arbitrary forces and constraints to either forces without constraints or constraints without forces. Moreover, by 'deepening the foundations', one may arrive at a physically unintuitive formulation, given that one intends to keep a very deep mathematical concept, such as continuity.

> The axioms of classical mechanics can be deepened if, using the axiom of continuity, one imagines continuous motions to be decomposed into small uniform rectilinear motions caused by discrete impulses and following one another in rapid succession. One then applies Bertrand's maximum principle as the essential axiom of mechanics, according to which the motion that actually occurs after each impulse is that which always maximizes the kinetic energy of the system with respect to all motions that are compatible with the law of the conservation of energy (1918, p. 409/1111).

All four requirements together with the fact that a unique and realist interpretation was not aspired at, suggest considering Hilbert's axiomatic method as a mathematical reorganization and stratification of a physical theory aimed at casting as much as possible in terms of constitutive a priori principles. For, mathematics has the advantage that the relationship across frameworks is rigorous and that one can precisely spot coordinating principles, such as general covariance and – less convincingly – Bertrand's principle. Moreover, Hilbert's axiomatic treatment of phenomenological theories, among them Kirchhoff's law of radiation and continuum mechanics, shows that he tried to establish empirical facts as constitutive principles under the joint guidance of a general philosophical outlook on the axiomatization of science and of the well-known heuristic powers of the PLA. This did not involve realist or reductionist aspirations, especially after the road to an immutable foundation of mathematics was barred. In the case continuum mechanics, he simply argued as had Planck in the above mentioned case of radiation, that the PLA was applicable even though, as of 1907, knowledge about the molecular constitution of matter was insufficient.

But there are problems to such an interpretation, especially if one looks at the "Foundations". For one, Hilbert's first axiom already contained a strong physical claim insofar as all energy-matter was subsumed under Mie's theory, a claim that Einstein considered as highly premature. Even more, the whole rationale of the failed theorem I was to blur the boundary between mathematics and physics. For, Hilbert believed "that a reduction of all physical constants to mathematical constants should be possible" (1916, p. 407). Hence, in some cases, the 'deepening' was to unearth what Hilbert typically – and pretty vaguely – described as the non-Leibnizian pre-established harmony between mathematics and physics. Not least this repeated allusion made Hilbert's axiomatic method suspicious among Logical Empiricists.

But the problem of non-uniqueness is more generic. The mathematician's 'deepening' and the physicist's search for the 'deepest' principles may yield diverging results. This is problematic even if one does not heed realist or structural realist aspirations. What is more, throughout its history the PLA has always been accompanied by largely equivalent formulations in terms of differential equations. Even if one emphasizes the philosophical and mathematical differences between both approaches – as I have done in the present paper – there is little sign of convergence to constitutive principles "of ever greater generality and adequacy" (Friedman, 2001, p. 63).

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